ANALYTIC RESULT FOR THE PROPERTIES OF GRAVITATIONAL WAVES EMITTED BY A LARGE CLASS OF MODEL SOURCES

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Abstract. Expressions for the mass quadrupole moment tensor in the model for a wide variety of astrophysical objects are shown to be identical in form. This makes it possible to obtain analytical expressions for the gravitational radiation emitted by the sources, as well as the angular distribution, polarization dependence, and the wave forms of the radiation.

1. Introduction

With the expected rapid advance in detector technology for gravitational waves (Amaldi, 1980; Blair, 1980; Lipa, 1980; Richard, 1980; Weber and Hirakawa, 1981; Hellings, 1981; Smarr *et al.*, 1983; Shapiro and Teukolsky, 1983) it becomes increasingly important to have reliable calculations not just for the energy radiated, but also for the angular/polarization dependence and wave forms. The latter would provide detailed information on the dynamics and symmetry of the source and can also be used to discriminate between rival theories of gravitation, obviously reasons behind the suggestion of Thorne and Braginski (1976) for a comprehensive cataloguing of wave forms for various astrophysical sources.

Instead of carrying out the calculation for a specific case, we find that for a large class of realistic situations, the expressions for the mass quadrupole tensor all have the *same*, relatively simple, form (see Section 2). We were able then to obtain a simple, *analytic*, formula for the radiation from these sources, as well as explicit expressions for the radiation pattern, polarization dependence and the wave form. This should be useful in considerations for detector design and identification of the sources under study.

2. Formulation

Consider a source of gravitational radiation characterized by a mass quadrupole moment tensor $D_{\alpha\beta}$ of the form

$$(D_{\alpha\beta}) = \begin{pmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{pmatrix} \tag{1}$$

with respect to a set of fixed inertial axes (x_1, x_2, x_3). In an actual physical setting, the

x_3-direction could assume the invariant one of the angular momentum or the rotation. No such physical meaning is assigned to the x_1- and x_2-axes. $D_{\alpha\beta}$ is defined by

$$D_{\alpha\beta} = \int \rho(\mathbf{x})\,(3x_\alpha x_\beta - x_\gamma^2\,\delta_{\alpha\beta})\,\mathrm{d}\mathbf{x}\,, \tag{2}$$

where summation over repeated indices is understood hereinafter (unless otherwise stated) and the various symbols have their usual meaning. It is obvious from Equation (2) that $D_{\alpha\beta} = D_{\beta\alpha}$ and $D_{\alpha\alpha} \equiv 0$.

The waves can be taken to be plane in view of the typically large distance between the source and the observer. The two independent polarization states can then be best represented in the Transverse Traceless (TT) gauge (see, e.g., Misner *et al.*, 1973) by the three-dimensional symmetric, unit polarization tensor $e_{\alpha\beta}$ satisfying the relations

$$e_{\alpha\alpha} = 0\,, \qquad e_{\alpha\beta} n_\beta = 0\,, \qquad e_{\alpha\beta} e_{\alpha\beta} = 1\,, \tag{3}$$

where $\mathbf{n}$ is a unit vector in the direction of propagation of the wave. The expressions for the intensity of radiation of a given polarization into solid angle $\mathrm{d}\Omega$ can then readily obtained (Landau and Lifshitz, 1975)

$$\mathrm{d}I = \frac{G}{72\pi c^5}\,(\dddot{D}_{\alpha\beta} e_{\alpha\beta})^2\,\mathrm{d}\Omega \quad \text{(no summation)}\,. \tag{4}$$

The total intensity for all polarization states can be obtained by taking the sum of Equation (4) applied to the different polarization or by using the alternative, but completely equivalent expression

$$\mathrm{d}I = \frac{G}{32\pi c^5}\,[\tfrac{1}{4}(\dddot{D}_{\alpha\beta} n_\alpha n_\beta)^2 + \tfrac{1}{2}\dddot{D}_{\alpha\beta}^2 - \dddot{D}_{\alpha\beta}\dddot{D}_{\alpha\gamma} n_\beta n_\gamma]\,\mathrm{d}\Omega\,. \tag{5}$$

The total rate of radiation in *all* directions, then, can be obtained by simply integrating Equation (5), over all directions, leading to the familiar result

$$-\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{G}{45c^5}\,(\dddot{D}_{\alpha\beta})^2\,. \tag{6}$$

The wave forms of the radiation received by an observer at distance r ($\gg$ dimension of the source) depend of course on the relative (angular) orientation between the observer and the source. In general, for an observer specified by the vector $\mathbf{r}$ making an angle θ_0 with the x_3-axis and ϕ_0 with the x_1-axis, the two polarization states can be characterized by the following expressions for the two non-vanishing components of the perturbation to the galilean metric (cf. Landau and Lifshitz, 1975)

$$h_\times = h_{\theta_0\phi_0} = -\frac{2G}{3c^4 r}\,\ddot{D}_{\theta_0\phi_0}\,, \tag{7}$$

$$h_+ = h_{\theta_0\theta_0} = -h_{\phi_0\phi_0} = \frac{-G}{3c^4 r}(\ddot{D}_{\theta_0\theta_0} - \ddot{D}_{\phi_0\phi_0}), \tag{8}$$

where $D_{\theta_0\phi_0}$, $D_{\theta_0\theta_0}$; and $D_{\phi_0\phi_0}$ are the physical components of D_{ij} projected along the directions of the spherical unit vectors $\mathbf{e}_{\theta_0}$ and $\mathbf{e}_{\phi_0}$. There exists canonical procedure for obtaining these components (see, e.g., Synge and Schild, 1949) and we simply quote here the results: i.e.,

$$D_{\theta_0\phi_0} = \cos\theta_0[\tfrac{1}{2}\sin 2\phi_0(D_{22} - D_{11}) + \cos 2\phi_0 D_{12}], \tag{9}$$

$$D_{\theta_0\theta_0} = \cos^2\theta_0(D_{11}\cos^2\phi_0 + D_{22}\sin^2\phi_0 + D_{12}\sin 2\phi_0) + D_{33}\sin^2\theta_0; \tag{10}$$

and

$$D_{\phi_0\phi_0} = D_{11}\sin^2\phi_0 + D_{22}\cos^2\phi_0 - D_{12}\sin 2\phi_0. \tag{11}$$

With the expressions given by Equations (9)–(11) we can readily obtain the explicit formulae for the intensity of radiation in the two polarization states as

$$\frac{\mathrm{d}I_1}{\mathrm{d}\Omega} = \frac{G}{36\pi c^5}\{\tfrac{1}{2}(\ddot{D}_{22} - \ddot{D}_{11})\sin 2\phi_0 + \ddot{D}_{12}(\cos^2\phi_0 - \sin^2\phi_0)\}^2\cos^2\theta_0 \tag{12}$$

and

$$\frac{\mathrm{d}I_2}{\mathrm{d}\Omega} = \frac{G}{144\pi c^5}\{\ddot{D}_{11}(\cos^2\theta_0\cos^2\phi_0 - \sin^2\phi_0) + \ddot{D}_{22}(\cos^2\theta_0\sin^2\phi_0 - \cos^2\phi_0) + \\ + \ddot{D}_{12}\sin 2\phi_0(1 + \cos^2\theta_0) + D_{33}\sin^2\theta_0\}^2. \tag{13}$$

In those cases where the system is rapidly rotating about the x_3-axis, it is appropriate to average over ϕ_0 (equivalently, one rotation period) when we have

$$\left\langle\frac{\mathrm{d}I_1}{\mathrm{d}\Omega}\right\rangle = \frac{G}{72\pi c^5}[\tfrac{1}{4}(\ddot{D}_{22} - \ddot{D}_{11})^2 + \ddot{D}_{12}^2]\cos^2\theta_0, \tag{14}$$

$$\left\langle\frac{\mathrm{d}I_2}{\mathrm{d}\Omega}\right\rangle = \frac{G}{144\pi c^5}\{\cos^4\theta_0[\tfrac{3}{8}(\ddot{D}_{11}^2 + \ddot{D}_{22}^2 + \tfrac{1}{2}\ddot{D}_{12}^2 + 2\ddot{D}_{33}^2 + \tfrac{1}{4}\ddot{D}_{11}\ddot{D}_{22})] + \\ + \cos^2\theta_0[\ddot{D}_{12}^2 - \tfrac{1}{4}(\ddot{D}_{11}^2 + \ddot{D}_{22}^2) - 4\ddot{D}_{33}^2 - \tfrac{3}{2}\ddot{D}_{11}\ddot{D}_{22}] + \\ + \tfrac{3}{8}(\ddot{D}_{11}^2 + \ddot{D}_{22}^2) + \tfrac{1}{2}\ddot{D}_{12}^2 + 2\ddot{D}_{33}^2 + \tfrac{1}{4}\ddot{D}_{11}\ddot{D}_{22}\}. \tag{15}$$

We have verified that the sum of Equations (14) and (15) gives a result identical to that obtained from Equation (5), as it should.

3. Application

A large class of realistic astrophysical systems turns out to have a $D_{\alpha\beta}$ of the form given in Equation (1). These include, for example, binary systems, rotating ellipsoidal objects and pulsating/rotating ellipsoids. The latter case, for large amplitude pulsation, corresponds of ocurse to explosion and collapse, and requires tedious numerical computation even though the basic formulae are the ones given in Section 2. It has been discussed elsewhere in detail (Saenz and Shapiro, 1979; Beltrami and Chau, 1984) and will, therefore, not be considered here.

3.1. Angular and Polarization Dependence

Equations (14) and (15) were applied to the case of a self-gravitating two-body system and the results of Landau and Lifshitz (1975) were recovered, thus offering a control check on the accuracy of these expressions.

For the case of a rigid, uniform density ellipsoid of mass M and semi-major axes a_1, a_2, a_3 rotating with an angular velocity ω about the x_3-axis, we have (cf. Chau, 1967):

$$D_{11} = D'_{11}\cos^2\omega t + D'_{22}\sin^2\omega t\,, \qquad D_{22} = D'_{11}\sin^2\omega t + D'_{22}\cos^2\omega t\,,$$
$$D_{12} = \tfrac{1}{2}\sin 2\omega t(D'_{11} - D'_{22})\,, \qquad D_{33} = D'_{33}\,; \tag{16}$$

where the D'_{ij}'s are the quadrupole moment tensor in the body axes of the ellipsoid, given by

$$D'_{ii} = \tfrac{1}{5}M(3a_i^2 - a_\alpha a_\alpha) \tag{17}$$

and

$$D'_{ij} = 0 \quad \forall\, i \neq j\,.$$

With Equations (16) and (17), from Equations (14) and (15) we readily obtain

$$\frac{\mathrm{d}I_1}{\mathrm{d}\Omega} = \frac{GM^2\omega^6}{50\pi c^5}\,(a_2^2 - a_1^2)^2\, 4\cos^2\theta_0\,, \tag{18}$$

and

$$\frac{\mathrm{d}I_2}{\mathrm{d}\Omega} = \frac{GM^2\omega^6}{50\pi c^5}\,(a_2^2 - a_1^2)^2\,(1 + 2\cos^2\theta_0 + \cos^4\theta_0)\,. \tag{19}$$

The intensity I over all polarization is, of course, given by

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{\mathrm{d}I_1}{\mathrm{d}\Omega} + \frac{\mathrm{d}I_2}{\mathrm{d}\Omega}\,;$$

which yields the following for the total energy radiated per sec

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int \frac{\mathrm{d}I}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = \frac{32}{125}\,\frac{GM^2\omega^6}{c^5}\,(a_2^2 - a_1^2)^2\,, \tag{20}$$

in agreement with known results (Chin, 1965; Chau, 1967). It is also obvious that $\mathrm{d}I_1/\mathrm{d}\Omega = \mathrm{d}I_2/\mathrm{d}\Omega \equiv 0$ for $a_1 = a_2$ as was to be expected.

For an incompressible spherical fluid mass of density ρ and radius R undergoing small amplitude axisymmetric oscillation, we can describe the boundary surface by (Chau, 1967)

$$r(\theta) = R(1 + \alpha_2 P_2(\theta) + \cdots + \alpha_n P_n(\theta)), \tag{21}$$

where the α_n's are functions of time and the P_n's are Legendre polynomials. The D_{ij}'s are readily calculated,

$$D_{ij} = 0 \quad \forall i \neq j, \qquad D_{11} = D_{22} = -\tfrac{1}{2}D_{33}\,; \tag{22}$$

with

$$D_{33} = \frac{8\pi\rho R^5}{5}\left[\alpha_2 + \sum_{n=2} \frac{10\alpha_n^2 n(n+1)}{(2n-1)(2n+1)(2n+3)} + \right.$$

$$\left. + \sum_{n=2} \frac{30\alpha_n\alpha_{n+2}(n+1)(n+2)}{(2n+1)(2n+3)(2n+5)} + \cdots\right].$$

This then yields

$$\frac{\mathrm{d}I_1}{\mathrm{d}\Omega} = 0 \tag{23}$$

and

$$\frac{\mathrm{d}I_2}{\mathrm{d}\Omega} = \frac{9}{8}\,\frac{G}{72\pi c^5}\,(\dddot{D}_{33})^2\,(1 - 2\cos^2\theta_0 + \cos^4\theta_0)\,. \tag{24}$$

Equation (24) when integrated over all angles again gives us back the expected expression for $\mathrm{d}E/\mathrm{d}t$ (Chau, 1967). We remark that all the radiation is in only one polarization mode, which is *true* for all axisymmetric systems, as is obvious from Equation (14). Furthermore, for such systems, the angular distribution is always of the form $(1 - 2\cos^2\theta_0 + \cos^4\theta)$, as is also obvious from Equation (24).

For the more interesting case of a uniform density (ρ) sphere of radius R and mass m, undergoing pulsation with a zero-th order (radial mode) frequency σ_0, but perturbed by a rotation Ω, we have (Chau, 1967)

$$D_{33} = -\frac{8}{15}\,\pi\rho\alpha R^5 \cos\sigma_0 t\left(\frac{45}{4}\gamma - \frac{9}{5}\right)\frac{\Omega^2}{\sigma_0^2}\,; \tag{25}$$

where γ is the ratio of the specific heats and α is a dimensionless constant characterizing the Lagrangian displacement $\xi_r = \alpha r \exp(i\sigma_0 t)$.

By use of (25), the expressions for the angular distribution of the energy in the different polarization modes can again be readily obtained. Thus we have

$$\frac{\mathrm{d}I_1}{\mathrm{d}\Omega} = 0 \tag{26}$$

and

$$\frac{dI_2}{d\Omega} = \frac{9}{7200}\,\frac{m^2\alpha^2R^4\sigma_0^6}{\pi c^5}\left(\frac{45}{4}\gamma - \frac{9}{5}\right)^2\left(\frac{\Omega^2}{\sigma_0^2}\right)^2 \times$$
$$\times (1 - 2\cos^2\theta_0 + \cos^4\theta_0)\,. \tag{27}$$

An integration over all angles yields an expression for $(\mathrm{d}E/\mathrm{d}t)$ that agrees, as it should, with the corresponding ones given by Chau (1967).

3.2. Wave forms

The expressions for the wave forms can be worked out explicitly for a given source with the aid of Equations (7) and (8) for $h_\times$ and h_+, together with Equations (9)–(11) for the $D_{\alpha\beta}$'s. We make use of the freedom of rotation about the x_3-axis so that the observer is on the $x_1 - x_3$ plane, i.e., ϕ_0 can be set equal to 0. We then readily obtain from Equations (7)–(11) the relations

$$h_\times = h_{\theta_0\phi_0} = -\frac{2G}{3rc^4}\,\dddot{D}_{12}\cos\theta_0\,, \tag{28}$$

$$h_+ = h_{\theta_0\theta_0} = -h_{\phi_0\phi_0} = \frac{-G}{3rc^4}\,[(\dddot{D}_{22} - \dddot{D}_{11}) + (\dddot{D}_{11} - \dddot{D}_{33})\sin^2\theta_0]\,. \tag{29}$$

For two-point masses m_1, m_2 in circular orbit of radius a and frequency ω, we have

$$h_\times = \frac{-4G}{rc^4}\,\mu a^2\omega^2\sin 2\psi\cos\theta_0 \tag{30}$$

and

$$h_+ = -\frac{2G}{rc^4}\,\mu a^2\omega^2\cos 2\psi(1 + \cos^2\theta_0)\,, \tag{31}$$

where μ is the reduced mass, and $\mathrm{d}\psi/\mathrm{d}t = \omega$.

For the rotating ellipsoid discussed in Section 3.1, we have

$$h_\times = -\frac{4G}{5rc^4}\,M\omega^2(a_2^2 - a_1^2)\sin 2\psi\cos\theta_0 \tag{32}$$

and

$$h_+ = -\frac{2G}{5rc^4}\,M\omega^2(a_1^2 - a_2^2)\cos 2\psi(1 + \cos^2\theta_0)\,. \tag{33}$$

For the non-radially pulsating sphere, or indeed for any axisymmetric system, we have

$$h_\times \equiv 0 \tag{34}$$

and

$$h_+ = -\frac{G}{2rc^4}\,\ddot{D}_{33}\sin^2\theta_0\,, \tag{35}$$

with D_{33} given by Equation (22).

Equations (34) and (35) are of course also valid for the case of the rotationally perturbed, radially pulsating sphere discussed in Section 3.1, with D_{33} given by Equation (25).

4. Conclusions

We have shown that a large class of model sources believed to be efficient emitters of gravitational waves has a mass quadrupole moment that readily admits analytic expressions for the angular distributions, polarization dependence and wave amplitudes. The expressions were worked out explicitly for several realistic, astrophysical systems and add to the catalogue of wave form calculations considered useful for gravitational wave astronomy.

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